Jan 3 2017, 15:00 (Duration: 100 min)

MATH 101/1001 Calculus I Final Exam

Name Surname: ____________________________**Signature:**________________

Department: _______________________________**Student Number:**__________

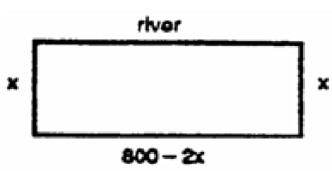
In solving the following problems, you are required to show all your work and provide the necessary explanations everywhere to get full credit.

Problem 1: [15 pts] Evaluate $\lim_{x \to 0} \frac{x - sinx}{x - tanx}$

Solution: $\lim_{x \to 0} \frac{x - sinx}{x - tanx} = \lim_{x \to 0} \frac{1 - cosx}{1 - sec^2 x} = -\frac{1}{2}$

Problem 2: [15 pts] A rectangular field will be bounded on one side by a river and on the other three sides by an electric fence. With 800 m of wire available, what is the largest area you can enclose and what are its dimensions?

The area is $A(x) = x(800 - 2x)$, where $0 \le x \le 400$. Solving $A'(x) = 800 - 4x = 0 \Rightarrow x = 200$. With $A(0) = A(400) = 0$, the maximum area is $A(200) = 80,000 \text{ m}^2$. The dimensions are 200 m by 400 m.



Problem 3: [30 pts]

Evaluate the following integrals:

a) $\int \frac{x}{\sqrt{16-x^2}} dx$

Solution: Let $x = 4sin\theta$. Then $dx = 4cos\theta d\theta$.

$\int \frac{x}{\sqrt{16-x^2}} dx = \int \frac{4sin\theta 4cos\theta}{\sqrt{16(1-sin^2\theta)}} d\theta = 4 \int sin\theta d\theta = -4cos\theta + c = -\sqrt{16 - x^2}$+c

b) $\int \frac{3}{x^3+2x^2+x} dx$

Solution: We do partial fractions: $\frac{3}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.

We get $A = 3, B = -3, C = -3$. Then integral becomes

$$\int \frac{3}{x^3 + 2x^2 + x} dx = \int (\frac{3}{x} - \frac{3}{x+1} - \frac{3}{(x+1)^2}) dx$$

$$= 3ln|x| - 3ln|x+1| + \frac{3}{x+1} + c$$

Problem 4: [20 pts]

a) Evaluate $\int \frac{e^x}{e^x-1} dx$

b) Determine whether the following improper integral is convergent or divergent:

$$\int_0^1 \frac{e^x}{e^x-1} dx$$

Solution: $\int_0^1 \frac{e^x}{e^x-1} dx = \lim_{c \to 0^+} \int_c^1 \frac{e^x}{e^x-1} dx$. Letting $e^x - 1 = u$ first, $\int \frac{e^x}{e^x-1} dx = ln|e^x - 1| + c$. So,

$$\int_0^1 \frac{e^x}{e^x-1} dx = \lim_{c \to 0^+} \int_c^1 \frac{e^x}{e^x-1} dx = \lim (ln|e^1 - 1| - ln|e^c - 1|) = +\infty$$

Hence, integral diverges.

Problem 5: [20 pts] Let R be a region bounded by the curves $y = sinx$, $y = cosx$ and x-axis in the first quadrant. Form the definite integrals giving the following but DO NOT evaluate:

a) Perimeter of the region R
b) Area of the region R
c) Volume of the solid obtained by revolving R about the x-axis
d) Volume of the solid obtained by revolving R about the y-axis.

Solution:

a) $\int_0^{\frac{\pi}{4}} \sqrt{1 + cos^2 x}dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 + sin^2 x}dx + \int_0^{\frac{\pi}{2}} \sqrt{1 + 0^2}\, dx$

b) $\int_0^{\frac{\pi}{4}} cosx\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sinx\, dx$

c) $\int_0^{\frac{\pi}{4}} \pi sin^2 xdx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \pi cos^2 xdx$ OR $\int_0^{\frac{\sqrt{2}}{2}} 2\pi y(arccosy - arcsiny)dy$

d) $\int_0^{\frac{\sqrt{2}}{2}} \pi(arccos^2 y - arcsin^2 y)dy$ OR $\int_0^{\frac{\pi}{4}} 2\pi x\, sinxdx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\pi x\, cosxdx.$